NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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INVESTIGATIONS ON THE STABILITY, OSCILLATION, AND STRESS CONDITIONS OF AIRPLANES WITH TAB CONTROL

FIRST PARTIAL REPORT - DERIVATION OF THE EQUATIONS OF MOTION AND THEIR GENERAL SOLUTIONS

By B. Filzek

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ABSTRACT: For the design and the construction of airplanes the control is of special significance, not only with regard to the flight-mechanical properties but also for the proportional arrangement of wing unit, fuselage, and tail unit. Whereas these problems may be regarded as solved for direct control of airplane motions, that is, for immediate operation of the control surfaces, they are not clarified as to oscillations, stability, and stress phenomena occurring in flight motions with indirect control, as realized for instance in tab control. Its modus operandi is based on the activation of a tab hinged to the trailing edge of the main control surface. Due to lift and drag variations, moments originate about the axis of rotation of the main control surface which cause an up-or-down floating of the main control surface and thus a change in the direction of the airplane. Since this tab control means flying with "free control surface", the treatment of this problem should provide the basis on which to judge stability, oscillation, and stress data.

Aside from the discussion of free oscillations and forced oscillating conditions, this report points out methods to deal, above all, with starting conditions. The required mathematical expedients are suitable, not only for the solution of the given problem, but also of similar problems, and are therefore treated in general formulation.

The present report is to represent a contribution toward the clarification of the problems arising and, first of all, to treat the longitudinal motion of an airplane.

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*"Untersuchungen über die Stabilitäts-, Schwingungs- und Beanspruchungsverhältnisse von Flugzeugen mit Hilfsrudersteuerung. 1. Teilbericht. Herleitung der Bewegungsgleichungen und ihre allgemeinen Lösungen." Zentrale für wissenschaftliches Berichtswesen der Luftfahrtforschung (ZWB), Forschungsbericht Nr. 2000, October 24, 1944. This translation is the first partial report (1. Teilbericht) of an investigation made up of two parts, the second part, FB 2000/2 (2. Teilbericht), of which is NACA TM 1198.

I. INTRODUCTION

The control of airplanes up to flying weights of about 20 tons is possible with direct control, that is, direct operation of the control surfaces by the pilot. However, for the modern large aircraft of about 30 tons flying weight and still more for the large aircraft to be expected in the future, the power required for direct activation of the control surfaces increases so greatly that human muscle power is no longer sufficient. One is obliged to use indirect controls, the necessary power being supplied for instance by a servomotor. The tendency will be to manage with a minimum of power.

The present report is to represent a contribution toward clarification of the problems arising and, above all, to treat the longitudinal motion of an airplane.

II. SYMBOLS

The symbols used correspond to the German standards DIN L 100 second edition, July 1939; moreover, the following symbols are chosen (cf. also fig. 1):

m	mass of the entire airplane
m_1	partial mass fuselage without control surfaces and tabs
m_2	partial mass control surfaces without tabs
m_3	partial mass of tabs
S	center of gravity of the whole system
S_1	center of gravity of the partial masses m_1
S_2	center of gravity of the partial masses m_2
S_3	center of gravity of the partial masses m_3
e_1	distance of the center of gravity of the partial mass m_1 from the center of gravity of the whole system
e_2	distance of the center of gravity of the partial mass m_2 from the center of rotation of the main control surface
e_3	distance of the center of gravity of the partial mass m_3 from the center of rotation of the tab
F_y	moment of inertia of the entire airplane about y-axis
F_{S_1}	moment of inertia of the partial mass m_1 about y-axis through S_1
F_{S_2}	moment of inertia of the partial mass m_2 about S_2
F_{S_3}	moment of inertia of the partial mass m_3 about S_3
F_{Hr}	moment of inertia of the elevator about axis of rotation of the main control surface
F_{Hh}	moment of inertia of the tab about axis of rotation of tab
η_h	deflection of the tab
$\Delta\eta_{hmax}$	deflection of the tab required for attaining the safe multiple of load
L_{Hr}	chord of the elevator + tab
L_{Hh}	chord of the tab
C_{MF}	coefficient of the air-force moment of the airplane without horizontal tail referred to center of gravity of the airplane
C_{nH}	coefficient of the normal force of the horizontal tail

C_{mH} coefficient of the horizontal tail moment referred to the axis through the aerodynamic center of the horizontal tail parallel to the y-axis

C_{rH} coefficient of the elevator moment referred to the axis of rotation of the main control surface

C_{rHh} coefficient of the tab moment referred to the axis of rotation of the tab

α_H angle of attack at the location of the horizontal tail

q_H mean dynamic pressure at the location of the horizontal tail surface

v_H mean velocity at the location of the horizontal tail surface

Δ mean downwash angle at the location of the horizontal tail

δ_H ratio of the air force damping of the entire airplane with respect to lateral axis to the air force damping of the horizontal tail

If similar symobls appear in the individual paragraphs (for instance, $\Delta\alpha$ = angle-of-attack increment, ΔM_L = moment increment, Δ = downwash angle, Δ_k = determinant, Δ^* = Routh's discriminant), they are distinguished by indices and their significance follows from the respective connection.

III. PRESUPPOSITIONS

In order to circumscribe the range of validity of this report, the necessary presuppositions shall be mentioned in advance; at the respective places they will be especially emphasized.

1. Unsteady air force influences are not taken into account.

2. Flight motions are possible only in the $X_g Z_g$-plane.

3. Method of small oscillations.

4. Fully compensated control surfaces, that is, the centers of gravity lie on the axes of rotation of the control surfaces.

5. Omission of the mass couplings and horizontal tail surface forces in the force equations.

6. Omission of the mass couplings and of the force couple at the horizontal tail at $C_{aH} = 0$ (zero moment) in the equation of moments about the lateral axis.

7. The velocity along the flight path v for the phenomenon under consideration is assumed to be constant.

8. Air forces and moments are assumed to be linear functions of their variables.

9. The steady state about which the system oscillates is assumed to be horizontal flight.

10. The dynamic pressure ratio q_H/q and the downwash factor $\partial\Delta/\partial\alpha$ are regarded as constant during the flight motions.

IV. DERIVATION OF THE EQUATIONS OF MOTION

For obtaining the equations of motion Lagrange's method will be used which permits the required equations of motion of a system with n degrees of freedom to be found from the following generally valid relation:

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_v}\right] - \frac{\partial T}{\partial q_v} = Q_v \tag{1}$$

Since this report deals with accelerated motions, it is useful to select for the determination of the kinetic energy T a nonaccelerated system of reference. For the present problem the coordinate system fixed to the ground may be regarded as such a system of reference. In that system, in its most general formulation, one would have to consider for an airplane in free flight the degrees of freedom

1. translation of the center of gravity of the airplane in X_g-direction,

2. translation of the center of gravity of the airplane in Y_g-direction,

3. translation of the center of gravity of the airplane in Z_g-direction,

4. rotation of the longitudinal inclination of the airplane about the center of gravity of the airplane ϑ,

5. rotation of the lateral inclination of the airplane about the center of gravity of the airplane φ,

6. rotation of the compass angle about the center of gravity of the airplane ψ,

7. rotation of the elevator about axis of rotation of the main control surface η.

However, for the consideration of the longitudinal motion of an airplane due to symmetric elevator operation one may assume that the airplane executes motions only in the X_gZ_g-plane. Thus no motions take place in the coordinates Y_g, φ, and ψ and the system is reduced to the degrees of freedom X_g,Z_g, ϑ, and η.

For the system sketched in figure 1 the kinetic energy assumes the form

$$\left.\begin{aligned} T = \Big[\frac{1}{2} F_{S_1}\dot{\vartheta}^2 &+ F_{S_2}(\dot{\vartheta} + \dot{\eta})^2 + F_{S_3}(\dot{\vartheta} + \dot{\eta} + \dot{\eta}_H)^2 \\ &+ m_1 v_{S_1}{}^2 + m_2 v_{S_2}{}^2 + m_3 v_{S_3}{}^2\Big] \end{aligned}\right\} \qquad (2)$$

The velocity coordinates v_{S_r} of the partial masses m_r result from the position coordinates X_{S_r}, Z_{S_r} by differentiation with respect to time as the relations

$$
\left.
\begin{aligned}
v_S{}^2 &= \dot{X}_g{}^2 + \dot{Z}_g{}^2 \\
v_{S_1}{}^2 &= \dot{X}_g{}^2 + \dot{Z}_g{}^2 + e_1{}^2\dot{\vartheta}^2 - 2e_1\dot{\vartheta}\left[\dot{X}_g\sin\vartheta + \dot{Z}_g\cos\vartheta\right] \\
v_{S_2}{}^2 &= \dot{X}_g{}^2 + \dot{Z}_g{}^2 + r_H{}^2\dot{\vartheta}^2 + e_2{}^2(\dot{\vartheta} + \dot{\eta})^2 + 2r_H\dot{\vartheta}\left[\dot{X}_g\sin\vartheta + \dot{Z}_g\cos\vartheta\right] \\
&\quad + 2e_2(\dot{\vartheta} + \dot{\eta})\left[\dot{X}_g\sin(\vartheta + \eta) + \dot{Z}_g\cos(\vartheta + \eta)\right] \\
&\quad + 2r_He_2\dot{\vartheta}(\dot{\vartheta} + \dot{\eta})\left[\sin\vartheta\sin(\vartheta + \eta) + \cos\vartheta\cos(\vartheta + \eta)\right] \\
v_{S_3}{}^2 &= \dot{X}_g{}^2 + \dot{Z}_g{}^2 + r_H{}^2\dot{\vartheta}^2 + \left(L_{Hr} - L_{Hh}\right)^2(\dot{\vartheta} + \dot{\eta})^2 + e_3{}^2\left(\dot{\vartheta} + \dot{\eta} + \dot{\eta}_H\right)^2 \\
&\quad + 2r_H\dot{\vartheta}\left[\dot{X}_g\sin\vartheta + \dot{Z}_g\cos\vartheta\right] \\
&\quad + 2\left(L_{Hr} - L_{Hh}\right)(\dot{\vartheta} + \dot{\eta})\left[\dot{X}_g\sin(\vartheta + \eta) + \dot{Z}_g\cos(\vartheta + \eta)\right] \\
&\quad + 2e_3\left(\dot{\vartheta} + \dot{\eta} + \dot{\eta}_h\right)\left[\dot{X}_g\sin\left(\vartheta + \eta + \eta_h\right) + \dot{Z}_g\cos\left(\vartheta + \eta + \eta_h\right)\right] \\
&\quad + 2r_H\left(L_{Hr} - L_{Hh}\right)\dot{\vartheta}(\dot{\vartheta} + \dot{\eta})\cos\eta + 2r_He_3\dot{\vartheta}\left(\dot{\vartheta} + \dot{\eta} + \dot{\eta}_h\right)\cos(\eta + \eta_h) \\
&\quad + 2e_3\left(L_{Hr} - L_{Hh}\right)(\dot{\vartheta} + \dot{\eta})\left(\dot{\vartheta} + \dot{\eta} + \dot{\eta}_h\right)\cos\eta_h
\end{aligned}
\right\} \quad (3)
$$

If the derivatives are formed according to the rule given in equation (1), one recognizes that the equations of motion are no longer linear and thus cannot be treated in a closed mathematical form. Therefore the expressions are already linearized at this point, that is, one puts $\cos X = 1$ and $\sin X = 0$.

These deliberations together with the summarizing of certain expressions yield for (2) the form

$$
\left.\begin{aligned}
T = \frac{1}{2}\Big\{ & m\left(\dot{X}_g^{\,2} + \dot{Z}_g^{\,2}\right) + F_y\dot{\vartheta}^2 + F_{Hr}\dot{\eta}^2 + F_{Hh}\dot{\eta}_h^{\,2} \\
& + 2\dot{\vartheta}\dot{\eta}\left[F_{Hr} + m_3r_H\left(L_{Hr} - L_{Hh}\right) + m_2e_2r_H + m_3e_3r_H\right] \\
& + 2\dot{\vartheta}\dot{\eta}_h\left[F_{Hh} + m_3e_3\left\{r_H + \left(L_{Hr} - L_{Hh}\right)\right\}\right] + 2\dot{\vartheta}\dot{Z}_g\left[r_H\left(m_2 + m_3\right) - e_1m_1\right] \\
& + 2\dot{\eta}\dot{\eta}_h\left[F_{Hh} + m_3e_3\left(L_{Hr} - L_{Hh}\right)\right] + 2(\dot{\vartheta} + \dot{\eta})\dot{Z}_g\left[m_3\left(L_{Hr} - L_{Hh}\right) + m_2e_2\right] \\
& + 2m_3e_3\left(\dot{\vartheta} + \dot{\eta} + \dot{\eta}_h\right)\dot{Z}_g\Big\}
\end{aligned}\right\} \quad (2a)
$$

If one introduces for the "generalized forces" Q_r convenient symbols and the derivatives $\frac{d}{dt}\left[\frac{\partial T}{\partial q_r}\right]$ into (1), the desired equations of motion read, if η_h is regarded not as free coordinate, but as disturbance function

$$
\left.\begin{aligned}
& \ddot{X}_g m = -A\sin\gamma - W\cos\gamma \\
& \ddot{Z}_g m + \ddot{\vartheta}\left[\left(m_2 + m_3\right)r_H + m_3\left(L_{Hr} - L_{Hh}\right) - m_1e_1 + m_2e_2 + m_3e_3\right] \\
& \qquad + \ddot{\eta}\left[m_3\left(L_{Hr} - L_{Hh}\right) + m_2e_2 + m_3e_3\right] = G + W\sin\gamma - A\cos\gamma \\
& \ddot{Z}_g\left[\left(m_2 + m_3\right)r_H + m_3\left(L_{Hr} - L_{Hh}\right) - m_1e_1 + m_2e_2 + m_3e_3\right] + \ddot{\vartheta}F_y \\
& \qquad + \ddot{\eta}\left[F_{Hr} + m_3r_H\left(L_{Hr} - L_{Hh}\right) + m_2e_2r_H + m_3e_3r_H\right] = M_L\left(\alpha, \alpha_H, \eta, \dot{\eta}, \eta_h\right) \\
& \ddot{Z}_g\left[m_3\left(L_{Hr} - L_{Hh}\right) + m_2e_2 + m_3e_3\right] + \ddot{\vartheta}\left[F_{H_r} + m_3r_H\left(L_{Hr} - L_{Hh}\right) + m_2e_2r_H \right. \\
& \qquad \left. + m_3e_3r_H\right] + \ddot{\eta}F_{Hr} = M_{Hr}\left(\alpha_H, \eta, \dot{\eta}, \eta_h\right)
\end{aligned}\right\} \quad (4)
$$

One sees that this system contains the eight variables X_g, Z_g, ϑ, γ, η, α, α_H, and η_h. Since there exist here four relations connecting the variables, which are given by

$$\left.\begin{aligned}
&\vartheta = \alpha + \gamma \\
&\alpha_H = \alpha_H(\alpha, \dot{\alpha}, \vartheta) \\
&\tan \gamma = \frac{\dot{Z}_g}{\dot{X}_g} \\
&\text{and} \\
&\eta_h = \eta_h(t) \qquad \text{(disturbance function)}
\end{aligned}\right\} \tag{5}$$

the four equations of motion derived in (4) are necessary and sufficient for a complete description of the problem.

The terms $m_r e_r$ and $r_H e_v m_v$, respectively, contained in this general formulation of the equations of motion permit – taking the weight moments appearing in M_L and M_{Hr} into consideration – the couplings caused by unmass-balanced control surfaces to be included.

An assumption of fully mass-balanced control surfaces, that is, $e_1 = e_2 = e_3 = 0$, does not only signify an essential simplification from the mathematical point of view; it is, with respect to e_2 and e_3, necessary to ensure freedom from flutter.

If one considers, moreover, the orders of magnitude of the individual sum terms in the corresponding equations, one finds that for rigid aircraft part of the coupling terms is of subordinate significance; if they are neglected, one obtains the following simplification of the equation system:

$$\left.\begin{aligned}
&\ddot{X}_g m = -A \sin \gamma - W \cos \gamma \\
&\ddot{Z}_g m = G + W \sin \gamma - A \cos \gamma \\
&\ddot{\vartheta} F_y = M_L \\
&\ddot{\vartheta}\left[F_{Hr} + m_3 r_H \left(L_{Hr} - L_{Hh}\right)\right] + \ddot{\eta} F_{Hr} = M_{Hr}
\end{aligned}\right\} \tag{6a}$$

or, written in the manner customary for flight mechanics,

$$\left.\begin{aligned}
m\frac{dv}{dt} &= -W - G\sin\gamma \\
mv\frac{d\gamma}{dt} &= A - G\cos\gamma \\
F_y\ddot{\vartheta} &= M_L \\
\ddot{\vartheta}\left[F_{Hr} + m_3 r_H\left(L_{Hr} - L_{Hh}\right)\right] &+ \ddot{\eta}F_{Hr} = M_{Hr}
\end{aligned}\right\}\tag{6b}$$

As shown by flight measurements and calculations,[1] the first equation of (6b) affects the variation of the coordinates γ and ϑ only slightly, due to the small change in speed v along the flight path at starting conditions. If one disregards, therefore, this equation for the solution, the remaining equations with $v = \text{const.} = v_o$ are sufficient for the description of the problem and one obtains the equations of motion

$$\left.\begin{aligned}
mv_o\frac{d\gamma}{dt} &= A - G\cos\gamma \\
F_y\ddot{\vartheta} &= M_L \\
\left[F_{Hr} + m_3 r_H\left(L_{Hr} - L_{Hh}\right)\right]\ddot{\vartheta} &+ F_{Hr}\ddot{\eta} = M_{Hr}
\end{aligned}\right\}\tag{6c}$$

the further mathematical investigations will be based on them in this shape.

The first equation of the equations of motion of (6c) deals with the forces perpendicular to the plane of the trajectory.

The second equation represents the condition for moment equilibrium about the center of gravity of the airplane, whereas the third equation expresses the condition for moment equilibrium about the axis of rotation of the main control surfaces.

For the "generalized forces" Q_r, designated in (6c) by A, M_L and M_{Hr}, one makes the assumption that they are linear functions of the variables. Then the expressions

$$A = \frac{\partial C_a}{\partial \alpha} qF\alpha \tag{7}$$

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[1]Bibliography (4)

$$\left.\begin{aligned} M_L &= \frac{\partial C_{MF}}{\partial \alpha} qFL\alpha - \left\{\frac{\partial C_{nH}}{\partial \alpha_H}\alpha_H + \frac{\partial C_{nH}}{\partial \eta}\eta + \frac{\partial C_{nH}}{\partial \eta_H}\eta_H\right\} F_H r_H q \frac{q_H}{q} \\ &+ \left\{\frac{\partial C_{mH}}{\partial \alpha_H}\alpha_H + \frac{\partial C_{mH}}{\partial \eta}\eta + \frac{\partial C_{mH}}{\partial \eta_H}\eta_H\right\} F_H L_H q \frac{q_H}{q} \end{aligned}\right\} \tag{8}$$

are valid for (6c,1) and (6c,2). A and M_L are, therefore, relatively independent of the method of control by tabs.

In expression of the external moment M_{Hr} in (6c,3), however, the form of the tab control obtains decisive importance.

In figures 2 and 3 the most frequently used tab controls are represented schematically.

The direct control by tabs is constructed so that no direct mechanical control possibility exists between control stick and tab. Instead, a deflection of the control stick actuates a servo-unit fixed to the main control surface and thus effects a displacement with respect to the main control surface.

For indirect tab control a deflection of the control stick actuates, by a push-pull control, a lever pivoting on the axis of rotation of the main control surface. Due to the relative motion of the lever with regard to the main control surface deflections of the tabs are effected. If one inserts, moreover, a spring between lever and main control surface, the result represents a combination of a servotab control and a direct control. If one makes the spring infinitely rigid, the system is transformed into a direct control by main control surfaces.

For the control by tabs visualized in figure 2 the expression for M_{Hr} reads

$$M_{Hr} = \left\{\frac{\partial C_{rH}}{\partial \alpha_H}\alpha_H + \frac{\partial C_{rH}}{\partial \eta}\eta + \frac{\partial C_{rH}}{\partial \eta_H}\eta_H\right\} F_H L_H q \frac{q_H}{q} - {}^{(1)}A_{1\eta}\eta \tag{9}$$

the factor ${}^{(1)}A_{1\eta}$ is to be interpreted as damping factor of the main control surface.

If one wants, on the other hand, to give the expression for an indirect tab control (fig.3), the relation

$$M_{Hr} = M_{CrH} + M_{Fed} + M_{Sth} + M_D$$

must be taken into consideration, with

M_{CrH} = airforce moment of the not moving elevator

M_{Fed} = moment of the spring

M_{Sth} = moment of the tab rod force

M_D = moment of the main control surface damping = $-{}^{(1)}A_{1\eta}\dot{\eta}$

If in figure 4

φ signifies the deflection of the driving lever a

η the deflection of the main control surface

η_H the deflection of the tab

i the mechanical gear ratio between tab deflection and driving lever a

there exists between the three angles, due to mechanical coupling, the relation

$$-\eta_H = (\varphi - \eta)i(\varphi,\eta) \tag{10}$$

Due to the deflection of the spring caused by the activation of the lever a, a moment of the following magnitude arises

$$M_{Fed} = C_{Fed}f(\varphi,\eta) \tag{11}$$

with C_{Fed} signifying the moment of the spring about the axis of rotation of the main control surface referred to 1°.

The tab push-rod force P_{Sth} produces a moment about the axis of rotation of the main control surface

$$M_{Sth} = M_{Hh}i(\varphi,\eta) \tag{12}$$

with

$$M_{Hh} = \left\{\frac{\partial C_{rHh}}{\partial \alpha_H}\alpha_H + \frac{\partial C_{rHh}}{\partial \eta}\eta + \frac{\partial C_{rHh}}{\partial \eta_H}\eta_H\right\}F_H L_H q\frac{q_H}{q} \tag{12a}$$

For small angles the gear ratio may be put as

$$i(\varphi,\eta) = \text{const.} = i \tag{10a}$$

and

$$f(\varphi,\eta) = (\varphi - \eta) \tag{11a}$$

Only variations with respect to a condition of steady motion are decisive for the course of the motion. Thus one will introduce for the variables the expression

$$\begin{aligned} \gamma &= \gamma_o + \Delta\gamma \\ \alpha &= \alpha_o + \Delta\alpha \qquad \text{etc.} \end{aligned}$$

One then obtains for instance for the first equation of (6c,1)

$$mv_o\frac{d\gamma}{dt} = \frac{\partial C_a}{\partial\alpha}q_oF(\alpha_o + \Delta\alpha) - G\cos(\gamma_o + \Delta\gamma)$$

and by subtraction of the steady starting condition

$$0 = \frac{\partial C_a}{\partial\alpha}q_oF\alpha_o - G\cos\gamma_o$$

the relation

$$mv_o\frac{d\gamma}{dt} = \frac{\partial C_a}{\partial\alpha}q_oF\Delta\alpha + G\sin\gamma_o\,\Delta\gamma \tag{13}$$

If one treats the expressions for M_L and M_{Hr} in a similar manner and takes into account that the coefficients C_{MF}, C_{nH}, C_{mH}, C_{rH} may be expressed as linear functions of the corresponding angles[2] for instance for

$$C_{rH} = \left[\frac{\partial C_{rH}}{\partial\alpha_H}\right]\alpha_H + \left[\frac{\partial C_{rH}}{\partial\eta}\right]\eta + \left[\frac{\partial C_{rH}}{\partial\eta_h}\right]\eta_h \tag{14}$$

their variations with respect to the steady condition assume the form

$$\left.\begin{aligned} \Delta M_L = {} & \frac{\partial C_{MF}}{\partial\alpha}qFL\Delta\alpha - \left\{\frac{\partial C_{nH}}{\partial\alpha_H}\Delta\alpha_H + \frac{\partial C_{nH}}{\partial\eta}\Delta\eta + \frac{\partial C_{nH}}{\partial\eta_h}\Delta\eta_h\right\}F_Hr_Hq\frac{q_H}{q} \\ & + \left\{\frac{\partial C_{mH}}{\partial\alpha_H}\Delta\alpha_H + \frac{\partial C_{mH}}{\partial\eta}\Delta\eta + \frac{\partial C_{mH}}{\partial\eta_h}\Delta\eta_h\right\}F_HL_Hq\frac{q_H}{q} \end{aligned}\right\} \tag{8a}$$

[2]Bibliography (5)

and

$$\Delta M_{Hr} = \left\{\frac{\partial C_{rH}}{\partial \alpha_H}\Delta\alpha_H + \frac{\partial C_{rH}}{\partial \eta}\Delta\eta + \frac{\partial C_{rH}}{\partial \eta_h}\Delta\eta_h\right\}F_H L_H q\frac{q_H}{q} - {}^{(1)}A_{1\eta}\dot{\eta} \tag{9a}$$

or, respectively, taking (10) to (12) into consideration,

$$\left.\begin{aligned}\Delta M_{Hr} = & \left\{\frac{\partial C_{rH}}{\partial \alpha_H} + i\frac{\partial C_{rHh}}{\partial \alpha_H}\right\}F_H L_H q\frac{q_H}{q}\Delta\alpha_H \\ & + \left\{\left[\left(\frac{\partial C_{rH}}{\partial \eta} + i\frac{\partial C_{rHh}}{\partial \eta}\right) + i\left(\frac{\partial C_{rH}}{\partial \eta_h} + i\frac{\partial C_{rHh}}{\partial \eta_h}\right)\right]F_H L_H q\frac{q_H}{q} - C_{Fed}\right\}\Delta\eta \\ & - \left\{i\left(\frac{\partial C_{rH}}{\partial \eta_h} + i\frac{\partial C_{rHh}}{\partial \eta_h}\right)F_H L_H q\frac{q_H}{q} - C_{Fed}\right\}\Delta\varphi - {}^{(1)}A_{1\eta}\dot{\eta}\end{aligned}\right\} \tag{9b}$$

the factor ${}^{(1)}A_{1\eta}$ of the last term of the equation (9b) is to be regarded as damping parameter of the main control surface.

For the angle of attack α_H at the location of the horizontal tail the expression customary in flight mechanics will be used.

$$\Delta\alpha_H = \left(1 - \frac{\partial\Delta}{\partial\alpha}\right)\Delta\alpha + \frac{r_H}{v_H}\frac{\partial\Delta}{\partial\alpha}\dot{\alpha} + \frac{r_H}{v_H}\delta_H\dot{\vartheta} \tag{15}$$

If in (13) the horizontal flight is regarded as starting condition, one obtains

$$\frac{d\gamma}{dt} = \frac{\partial C_a}{\partial\alpha}\frac{F}{G}g\sqrt{\frac{\rho}{2}q_o}\,\Delta\alpha = C_N\Delta\alpha$$

and from the relations connecting the variables (5) result – taking into consideration the expression

$$\vartheta = \vartheta_o + \Delta\vartheta$$

and

$$\dot{\vartheta} = \frac{d\vartheta}{dt} = \frac{d(\Delta\vartheta)}{dt} = \Delta\dot{\vartheta}$$

the relations

$$\left.\begin{aligned} \Delta\vartheta &= \Delta\alpha + \Delta\gamma \\ \dot{\vartheta} &= \dot{\alpha} + \dot{\gamma} = \dot{\alpha} + C_N\Delta\alpha \\ \ddot{\vartheta} &= \ddot{\alpha} + \ddot{\gamma} = \ddot{\alpha} + C_N\dot{\alpha} \end{aligned}\right\} \qquad (16)$$

With the according summarizations and transformations the relations (5) to (16) yield, inserted into (6c), in the conventional form the differential-equation system with the two degrees of freedom α and η

$$\left.\begin{aligned} \ddot{\eta} + {}^{(1)}A_{1\eta}\dot{\eta} + {}^{(0)}A_{1\eta}\Delta\eta + {}^{(2)}A_{1\alpha}\ddot{\alpha} + {}^{(1)}A_{1\alpha}\dot{\alpha} + {}^{(0)}A_{1\alpha}\Delta\alpha &= C_1\Delta S(t) \\ {}^{(0)}A_{2\eta}\Delta\eta + \ddot{\alpha} + {}^{(1)}A_{2\alpha}\dot{\alpha} + {}^{(0)}A_{2\alpha}\Delta\alpha &= C_2\Delta S(t) \end{aligned}\right\} \qquad (17)$$

one has to put for the variation of the disturbance function $\Delta S(t)$ in the case of direct tab operation

$$\Delta S(t) = \Delta\eta_h(t)$$

and in the case of indirect tab operation

$$\Delta S(t) = \Delta\varphi(t)$$

The differential equations thus obtained may now serve, for a prescribed dynamic pressure, for the determination of the course of the movement. Generally, however, it will be desired to treat the entire dynamic pressure range in a closed form. If one chooses, therefore, $\tau = \sqrt{q}\,t$ as new independent variable and puts $\Delta\eta = \Delta\eta(\tau)$; $\Delta\alpha = \Delta\alpha(\tau)$; $\Delta\eta_h = \Delta\eta_h(\tau)$ and, respectively, $\Delta\varphi = \Delta\varphi(\tau)$ one obtains

$$\left.\begin{aligned} \dot{\eta} &= \eta'\sqrt{q} \quad \text{and, respectively,} \quad \dot{\alpha} = \alpha'\sqrt{q} \\ \ddot{\eta} &= \eta''q \qquad\qquad\qquad\qquad\quad \ddot{\alpha} = \alpha''q \end{aligned}\right\} \qquad (18)$$

These relations (18) substituted into (17) permit, after division by q, a representation of the differential equations no longer explicitly dependent on the dynamic pressure which may then be written

$$\left.\begin{aligned} \eta'' + {}^{(1)}A_{1\eta}\eta' + {}^{(0)}A_{1\eta}\Delta\eta + {}^{(2)}A_{1\alpha}\alpha'' + {}^{(1)}A_{1\alpha}\alpha' + {}^{(0)}A_{1\alpha}\Delta\alpha &= C_1\Delta S(\tau) \\ {}^{(0)}A_{2\eta}\Delta\eta + \alpha'' + {}^{(1)}A_{2\alpha}\alpha' + {}^{(0)}A_{2\alpha}\Delta\alpha &= C_2\Delta S(\tau) \end{aligned}\right\} \quad (17a)$$

If one takes into consideration that the moment due to C_{mH} in (8) and (8a), respectively, remains small in comparison with the other quantities, the coefficients may be written

a) for direct tab control

$$\left.\begin{aligned}
{}^{(1)}A_{1\eta} &= {}^{(1)}A_{1\eta} \quad \text{parameter of the main-control-surface damping} \\
{}^{(0)}A_{1\eta} &= -\frac{\partial C_{rH}}{\partial\eta}\frac{F_H L_H}{F_{Hr}}\frac{q_H}{q} \\
{}^{(2)}A_{1\alpha} &= 1 + \frac{m_3 r_H\left(L_{Hr} - L_{Hh}\right)}{F_{Hr}} \\
{}^{(1)}A_{1\alpha} &= \left\{{}^{(2)}A_{1\alpha}\frac{\partial L_a}{\partial\alpha}\frac{F}{G}g - \frac{\partial C_{rH}}{\partial\alpha_H}\frac{F_H L_H r_H}{F_{Hr}}\frac{q_H}{q}\left(\delta_H + \frac{\partial\Delta}{\partial\alpha}\right)\right\}\sqrt{\frac{\rho}{2}} \\
{}^{(0)}A_{1\alpha} &= -\frac{\partial C_{rH}}{\partial\alpha_H}\frac{F_H L_H}{F_{Hr}}\left[1 - \frac{\partial\Delta}{\partial\alpha} + \frac{F}{G}g\frac{\rho}{2}\frac{\partial C_a}{\partial\alpha}r_H\delta_H\sqrt{\frac{q}{q_H}}\right]\frac{q_H}{q} \\
{}^{(0)}A_{2\eta} &= \frac{\partial C_{nH}}{\partial\eta}\frac{F_H r_H}{F_y}\frac{q_H}{q} \\
{}^{(1)}A_{2\alpha} &= \frac{\partial C_{nH}}{\partial\alpha_H}\left(\delta_H + \frac{\partial\Delta}{\partial\alpha}\right)r_H^2\frac{F_H}{F_y}\sqrt{\frac{\rho}{2}}\frac{q_H}{q} + \frac{\partial C_a}{\partial\alpha}\frac{F}{G}g\sqrt{\frac{\rho}{2}} \\
{}^{(0)}A_{2\alpha} &= \left[\frac{\partial C_{nH}}{\partial\alpha_H}\left\{1 - \frac{\partial\Delta}{\partial\alpha} + \frac{\partial C_a}{\partial\alpha}\frac{F}{G}g\frac{\rho}{2}r_H\delta_H\sqrt{\frac{q}{q_H}}\right\}F_H r_H\frac{q_H}{q} - \frac{\partial C_{MF}}{\partial\alpha}FL\right]\frac{1}{F_y} \\
C_1 &= -\frac{\partial C_{rH}}{\partial\eta_h}\frac{F_H L_H}{F_{Hr}}\frac{q_H}{q} \\
C_2 &= \frac{\partial C_{nH}}{\partial\eta_h}\frac{F_H r_H}{F_y}\frac{q_H}{q}
\end{aligned}\right\} \quad (17b)$$

b) for indirect tab control

$$
\left.
\begin{aligned}
{}^{(1)}A_{1\eta} &= {}^{(1)}\dot{A}_{1\eta} \qquad \text{parameter of the main-control-surface damping} \\
{}^{(0)}A_{1\eta} &= -\left\{\left[\frac{\partial C_{rH}}{\partial \eta} + i\frac{\partial C_{rHh}}{\partial \eta} + i\left(\frac{\partial C_{rH}}{\partial \eta_h} + i\frac{\partial C_{rHh}}{\partial \eta_h}\right)\right] F_H L_H \frac{q_H}{q} - \frac{C_{Fed}}{q}\right\}\frac{1}{F_{Hr}} \\
{}^{(2)}A_{1\alpha} &= 1 + \frac{m_3 r_H \left(L_{Hr} - L_{Hh}\right)}{F_{Hr}} \\
{}^{(1)}A_{1\alpha} &= \left\{{}^{(2)}A_{1\alpha}\frac{\partial C_a}{\partial \alpha}\frac{F}{G}g - \left(\frac{\partial C_{rH}}{\partial \alpha_H} + i\frac{\partial C_{rHh}}{\partial \alpha_H}\right)\frac{F_H L_H r_H}{F_{Hr}}\left(\delta_H + \frac{\partial \Delta}{\partial \alpha}\right)\sqrt{\frac{q_H}{q}}\right\}\sqrt{\frac{\rho}{2}} \\
{}^{(0)}A_{1\alpha} &= -\left\{\left(\frac{\partial C_{rH}}{\partial \alpha_H} + i\frac{\partial C_{rHh}}{\partial \alpha_H}\right)\left(1 - \frac{\partial \Delta}{\partial \alpha} + \frac{\partial C_a}{\partial \alpha}\frac{F}{G}g\frac{\rho}{2}r_H\delta_H\sqrt{\frac{q}{q_H}}\right)\frac{F_H L_H}{F_{Hr}}\frac{q_H}{q}\right. \\
{}^{(0)}A_{2\eta} &= \left\{\frac{\partial C_{mH}}{\partial \eta} + i\frac{\partial C_{nH}}{\partial \eta_h}\right\}\frac{F_H r_H}{F_y}\frac{q_H}{q} \\
{}^{(1)}A_{2\alpha} &= \left\{\frac{\partial C_{nH}}{\partial \alpha_H}\left(\delta_H + \frac{\partial \Delta}{\partial \alpha}\right)\frac{F_H r_H^2}{F_y}\sqrt{\frac{\rho}{2}}\frac{q_H}{q} + \frac{\partial C_a}{\partial \alpha}\frac{F}{G}g\sqrt{\frac{\rho}{2}}\right\} \\
{}^{(0)}A_{2\alpha} &= \left\{\frac{\partial C_{nH}}{\partial \alpha_H}\left(1 - \frac{\partial \Delta}{\partial \alpha} + \frac{\partial C_a}{\partial \alpha}\frac{F}{G}g\frac{\rho}{2}r_H\delta_H\sqrt{\frac{q}{q_H}}\right)F_H r_H\frac{q_H}{q} - \frac{\partial C_{MF}}{\partial \alpha}FL\right\}\frac{1}{F_y} \\
C_1 &= -\left\{i\left(\frac{\partial C_{rH}}{\partial \eta_h} + i\frac{\partial C_{rHh}}{\partial \eta_h}\right)F_H L_H\frac{q_H}{q} - \frac{C_{Fed}}{q}\right\}\frac{1}{F_{Hr}} \\
C_2 &= i\frac{\partial C_{nH}}{\partial \eta_h}\frac{F_H r_H}{F_y}\frac{q_H}{q}
\end{aligned}
\right\} \quad (17c)
$$

In the scope of this investigation it does not signify an essential limitation if one regards both dynamic pressure ratio q_H/q and downwash factor $\partial\Delta/\partial\alpha$ as constant not only during the flight motion but also for the dynamic pressure range to be investigated, so that with this presupposition a linear, inhomogeneous or complete system of differential equations with two degrees of freedom is presented for discussion.

V. GENERAL SOLUTION OF LINEAR DIFFERENTIAL EQUATION SYSTEMS (DES)

Knowledge of the solutions of inhomogeneous differential equations or, respectively, differential-equation systems is a presupposition for the treatment of starting phenomena as they are of importance in flight mechanics[3] and other technical fields. These solutions are given in the most general form for a linear DES with constant coefficients and n degrees of freedom with arbitrary disturbance functions and starting conditions.

1. Solution for Arbitrary Disturbance Functions and Starting Conditions

One assumes as given the DES with n degrees of freedom

$$\sum_{\kappa=1}^{n} \sum_{\mu=0}^{m} {}^{(\mu)}a_{\upsilon\kappa} X_\kappa^{(\mu)} = F_\upsilon(t) \tag{19}$$

Therein is:

κ = 1, 2, 3 ...... n the designation of the dependent variables

υ = 1, 2, 3 ...... n the notation of the independent differential equations

μ = 0, 1, 2 ...... m the order of derivative of the respective dependent variables

$X_\kappa^{(\mu)}$ = the μ-th derivative of the κ-th variable; $X_\kappa^{(0)}$ is to be $= X_\kappa$

${}^{(\mu)}a_{\upsilon\kappa}$ = the coefficient of the DES which stands for the μ-th derivative of the κ-th variable in the υ-th line.

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[3]Cf. bibliography, (4) and (6)

The desired solution of this problem can be given in a particularly clear form by using the Laplace transformation[4]. The required auxiliary theorems are introduced here without any proof.

If one applies to (19) the Laplace transformation

$$\mathcal{L}\left\{F(t)\right\} = \int_0^\infty F(\tau)C^{-p\tau}d\tau \tag{20}$$

and bears in mind that there exists the relation

$$\mathcal{L}\left\{F^{(\mu)}(t)\right\} = p^{\mu}\mathcal{L}\left\{F(t)\right\} - \sum_{\lambda=0}^{\mu-1} F^{(\lambda)}(0)p^{\mu-\lambda-1} \tag{21}$$

(with the sum to be put equal to zero for $\mu = 0$), (19) is transformed into

$$\sum_{\kappa=1}^{n}\sum_{\mu=0}^{m}{}^{(\mu)}a_{\upsilon\kappa}\left\{p^{\mu}\mathcal{L}\left[X_\kappa\right] - \sum_{\lambda=0}^{\mu-1} X_\kappa{}^{(\lambda)}(0)p^{\mu-\lambda-1}\right\} = \mathcal{L}\left\{F_\upsilon(t)\right\}$$

or

$$\sum_{\kappa=1}^{n} f_{\upsilon\kappa}(p)\mathcal{L}\left\{X_\kappa\right\} = \mathcal{L}\left\{F_\upsilon(t)\right\} + \sum_{\kappa=1}^{n} g_{\upsilon\kappa}(p) \tag{22}$$

with

$$f_{\upsilon\kappa}(p) = \sum_{\mu=0}^{m} {}^{(\mu)}a_{\upsilon\kappa}p^{\mu}$$ functions at most of m–th degree in p (23)

$$g_{\upsilon\kappa}(p) = \sum_{\mu=1}^{m}\sum_{\lambda=0}^{\mu-1} {}^{(\mu)}a_{\upsilon\kappa}X_\kappa{}^{(\lambda)}(0)p^{\mu-\lambda-1}$$ that is, functions at most (m – 1)–th degree in p (24)

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[4]Bibliography (8)

Therewith the linear DES with constant coefficients is reduced to an algebraic problem with the n unknown $\mathcal{L}\{X_\kappa\}$. The coefficients of the unknown $\mathcal{L}\{X_\kappa\}$ are rational integral functions at most of the m-th degree in p. The absolute terms on the right side of (22) consist of the Laplace transforms of the disturbance functions of the DES (19) and rational integral functions of at most (m − 1)-th degree in p which contain all starting conditions $X_\kappa^{(\lambda)}(0)$ with $\lambda = 0, 1, 2 \ldots\ldots (m-1)$ of the problem.

If one applies to (22) the theorems of the theory of determinants, the solution of the unknown $\mathcal{L}\{X_\kappa\}$ may in the most general form be written as follows:

$$\mathcal{L}\left\{X_\kappa\right\} = \frac{D_\kappa}{D} \tag{25}$$

therein D is the coefficient determinant, that is D = 0, the characteristic equation of the DES (19)

$$D(p) = \left|f_{\upsilon\kappa}(p)\right| \tag{26}$$

D_κ in each case the determinant which results from the coefficient determinant D if its κ-th column is replaced by the sequence of the absolute terms of (22)

$$\mathcal{L}\left\{F_\upsilon(t)\right\} + \sum_{\kappa=1}^{n} g_{\upsilon\kappa}(p) \tag{27}$$

Since, however, the determinant D_κ may be written as sum of two determinants one of which contains in the respective column the Laplace transforms of the disturbance functions $F_\upsilon(t)$, the second the sum $\sum_{\kappa=1}^{n} g_{\upsilon\kappa}(p)$, it will be useful to agree upon the following:

If in (22) all $F_\upsilon(t) = 0$ and at least one $X_\kappa^{(\lambda)}(0) \neq 0$ one is dealing with a homogeneous component solution; if at least one $F_\upsilon(t) \neq 0$ and all $X_\kappa^{(\lambda)}(0) = 0$, with an inhomogeneous one.

With this stipulation (25) may be written also

$$\mathcal{L}\left\{X_\kappa\right\} = \frac{D_{\kappa inh}}{D} + \frac{D_{\kappa hom}}{D} \tag{28}$$

If one now imagines the determinants written down and calculated, there is

1. D a polynomial of at most (mn)–th degree in p

$$D(p) = \sum_{\mu=0}^{mn} a_\mu p^\mu \tag{26a}$$

2. $D_{\kappa inh}$ a sum of polynomials of at most $m(n-1)$–th degree in p, since the κ–th column contains the sequence of the Laplace – transform $\mathcal{L}\left\{F_\upsilon(t)\right\}$. If this determinant $D_{\kappa inh}$ is developed with respect to the κ–th column, one can also write

$$D_{\kappa inh}(p) = \sum_{\upsilon=1}^{n} \mathcal{L}\left\{F_\upsilon(t)\right\} A_{\upsilon\kappa}(p) \tag{29}$$

with $A_{\upsilon\kappa}$ being the algebraic complement of the element $\mathcal{L}\left\{F_\upsilon(t)\right\}$ and a polynomial of at most $m(n-1)$–th degree in p.

3. $D_{\kappa hom}$ is a polynomial of at most $m(n-1)$–th degree in p

$$D_{\kappa hom}(p) = \sum_{\upsilon=1}^{n}\left\{\sum_{\kappa=1}^{n} g_{\upsilon\kappa}(p)\right\} A_{\upsilon\kappa}(p) = \sum_{\lambda=0}^{mn-1} {}^{(\lambda)}C_\kappa p^\lambda \tag{30}$$

so that after these deliberations (28) may be written in the form

$$\mathcal{L}\left\{X_\kappa\right\} = \sum_{\upsilon=1}^{n} \mathcal{L}\left\{F_\upsilon(t)\right\} \frac{A_{\upsilon\kappa}(p)}{D(p)} + \frac{D_{\kappa hom}(p)}{D(p)} \tag{28a}$$

with $A_{\upsilon\kappa}/D$ and $D_{\kappa hom}/D$ representing rational functions in the form of real fractions in p.

If the characteristic equation (26a) has only simple roots designated by $p_\mu(\mu = 0, 1, 2 \ldots\ldots (mn-1))$, the rational functions in the form of real fractions of the right side of (28) may be represented with the aid of decomposition to partial fractions in the following manner

$$\frac{A_{\upsilon\kappa}(p)}{D(p)} = \sum_{\mu=0}^{mn-1} \frac{A_{\upsilon\kappa}(p_\mu)}{D'(p_\mu)} \frac{1}{p - p_\mu} \tag{31}$$

and

$$\frac{D_{\kappa\,hom}(p)}{D(p)} = \sum_{\mu=0}^{mn-1} \frac{D_{\kappa\,hom}(p_\mu)}{D'(p_\mu)} \frac{1}{p - p_\mu} \tag{32}$$

therein

$$D'(p_\mu) = \left[\frac{dD}{dp}\right]_{p=p_\mu} = \prod_{\upsilon=0}^{mn-1}{}^{(\mu)}(p_\mu - p_\upsilon) \tag{33}$$

with the raised "(μ)" signifying that the factor is to be put equal to "1" for which $\upsilon = \mu$.

If one now introduces the two functions of time

$$G_{\upsilon\kappa}(t) = \sum_{\mu=0}^{mn-1} \frac{A_{\upsilon\kappa}(p_\mu)}{D'(p_\mu)} e^{p_\mu t} \tag{34}$$

and

$$H_\kappa(t) = \sum_{\mu=0}^{mn-1} \frac{D_{\kappa\,hom}(p_\mu)}{D'(p_\mu)} e^{p_\mu t} \tag{35}$$

their Laplace transforms will be exactly identical to the decompositions to partial fractions represented in (31) and (32).

Therewith (28) and (28a), respectively, are transformed into

$$\mathcal{L}\left\{X_\kappa\right\} = \sum_{\upsilon=1}^{n} \mathcal{L}\left\{F_\upsilon(t)\right\}\mathcal{L}\left\{G_{\upsilon\kappa}(t)\right\} + \mathcal{L}\left\{H_\kappa(t)\right\}$$

and, using the convolution [Faltungs] theorem

$$\mathcal{L}\left\{F(t)\right\}\mathcal{L}\left\{G(t)\right\} = \mathcal{L}\left\{F * G\right\} = \mathcal{L}\left\{\int_0^t F(\tau)G(t - \tau)d\tau\right\}$$

one obtains

$$\mathcal{L}\left\{X_\kappa\right\} = \mathcal{L}\left\{\sum_{\upsilon=1}^{n}\int_0^t F_\upsilon(\tau)G_{\upsilon\kappa}(t-\tau)d\tau + H_\kappa(t)\right\}$$

If one now cancels again the Laplace transformation and takes (34) and (35) into consideration, there results as solution of (19)

$$X_\kappa(t) = \sum_{\upsilon=1}^{n}\sum_{\mu=0}^{mn-1}\frac{A_{\upsilon\kappa}(p_\mu)}{D'(p_\mu)}e^{p_\mu t}\int_0^t F_\upsilon(\tau)e^{-p_\mu\tau}d\tau + \sum_{\mu=0}^{mn-1}\frac{D_{\kappa hom}(p_\mu)}{D'(p_\mu)}e^{p_\mu t} \qquad (36)$$

Thus, in the most general form, the solution of a linear DES with constant coefficients for arbitrary starting conditions and integrable disturbance functions is indicated in closed form.

It has to be noted that (36) was derived only under the assumption of simple roots p_μ of the characteristic equation $D(p) = 0$ of the DES.

If the solution for multiple roots of the characteristic equation $D(p) = 0$ also is of interest, no particular difficulty is encountered if the corresponding expressions of the related decompositions to partial fractions are taken into consideration and introduced.

2. Solution for Constant Disturbance Functions and Arbitrary Starting Conditions

For the practical engineer, however, this general solution gains full significance only when he is given ways and means to represent the desired functions X_κ numerically for his statement of the problem. The solutions X_κ may be regarded as known when one succeeds in representing, for analytically prescribed disturbance functions $F_\mu(t)$, the quadrature of the time integral $\int_0^t F_\upsilon(\tau)e^{-p_\mu\tau}d\tau$ in closed form.

In all cases where a representation in closed form is difficult or, respectively, the $F_\upsilon(t)$ are not prescribed analytically but are, for instance, known from tests as diagrams, there exists fundamentally the possibility of replacing the function $F_\upsilon(t)$ with a step function according to figure 2 in such a manner that the areas enclosed by the curve $F_\upsilon(t)$ or, respectively, by the step function and the abscissa are equal.

According to the mean value theorem of integral calculus, in an arbitrary time interval Δt

$$\int_{t_\mu}^{t_\mu+\Delta t} F_\upsilon(\tau)d\tau = \Delta t F_\upsilon(t_\xi) = \Delta t F_\upsilon(t_\mu + \vartheta\Delta t) \qquad 0 \leqq \vartheta \leqq 1$$

If one performs the limiting process with $\Delta t \rightarrow 0$, the step function tends toward the function $F_\upsilon(t)$ according to

$$\lim_{\Delta t \rightarrow 0} \frac{\int_{t_\mu}^{t_\mu + \Delta t} F_\upsilon(\tau)d\tau}{\Delta t} = F_\upsilon(t_\mu)$$

that is, one can approximate an arbitrarily prescribed integrable disturbance function $F_\upsilon(t)$ by according selection of the Δt with arbitrary accuracy.

One can therefore visualize any arbitrary disturbance phenomenon as built up from constant disturbance functions C_μ which commence to act at the times t_μ, so that one obtains a very serviceable approximation for the component solution $X_{\kappa inh}$ of the inhomogeneous DES by superposition of the partial solutions due to the constant disturbances C_μ.

For investigation of starting conditions the explicit form of (36) shall thus be given only for

$$F_\upsilon(t) = \text{Const.} = C_\upsilon$$

The inhomogeneous part of (36) then obtains the form

$$X_{\kappa inh} = \sum_{\upsilon=1}^{n} \sum_{\mu=0}^{mn-1} C_\upsilon \frac{A_{\upsilon\kappa}(p_\mu)}{D'(p_\mu)} \frac{e^{p_\mu t} - 1}{p_\mu} \tag{37}$$

which can still be considerably simplified.

If one forms, namely, from (31) the linear combination

$$\sum_{\upsilon=1}^{n} C_\upsilon \frac{A_{\upsilon\kappa}(p)}{D(p)} = \sum_{\upsilon=1}^{n} \sum_{\mu=0}^{mn-1} C_\upsilon \frac{A_{\upsilon\kappa}(p_\mu)}{D'(p_\mu)} \frac{1}{p - p_\mu} \tag{38}$$

the numerator of the left side of (38) represents a determinant which one obtains from the determinant of the characteristic equation (26) by replacing the κ-th column by the C_υ; thus one obtains a polynomial of at most $m(n - 1)$-th degree in p.

$$\sum_{\upsilon=1}^{n} C_\upsilon A_{\upsilon\kappa}(p) = \Delta_\kappa(p) = \sum_{\lambda=0}^{m(n-1)} {}^{(\lambda)}b_\kappa p^\lambda \tag{39}$$

For (38) one can therefore write abbreviatedly

$$\sum_{\upsilon=1}^{n} C_\upsilon \frac{A_{\upsilon\kappa}(p)}{D(p)} = \frac{\Delta_\kappa(p)}{D(p)} = \sum_{\mu=0}^{mn-1} \frac{\Delta_\kappa(p_\mu)}{D'(p_\mu)} \frac{1}{p - p_\mu} \tag{38a}$$

while (37) obtains the form

$$X_{\kappa inh} = \frac{\Delta_\kappa(0)}{D(0)} + \sum_{\mu=0}^{mn-1} \frac{\Delta_\kappa(p_\mu)}{p_\mu D'(p_\mu)} e^{p_\mu t} \tag{37a}$$

If one introduces, moreover, for the homogeneous component solution according to (30)

$$D_{\kappa hom}(p_\mu) = \sum_{\lambda=0}^{mn-1} {}^{(\lambda)}c_\kappa p_\mu{}^\lambda$$

and defines a function of time dependent only on the roots p_μ of the characteristic equation $D(p) = 0$

$$\varphi(t) = \sum_{\mu=0}^{mn-1} \frac{e^{p_\mu t}}{p_\mu D'(p_\mu)} \tag{40}$$

its λ-th derivative with respect to time is

$$\varphi^{(\lambda)}(t) = \sum_{\mu=0}^{mn-1} \frac{p_\mu^{\lambda-1}}{D'(p_\mu)} e^{p_\mu t} \tag{40a}$$

and the solution of linear DES with constant disturbance functions and arbitrary starting conditions obtains the very clearly arranged form

$$x_\kappa = \frac{\Delta_\kappa(0)}{D(0)} + \sum_{\lambda=0}^{m(n-1)} {}^{(\lambda)}b_\kappa \varphi^{(\lambda)}(t) + \sum_{\lambda=0}^{mn-1} {}^{(\lambda)}c_\kappa \varphi^{(\lambda+1)}(t) \tag{41}$$

whereas the derivative with respect to time of the κ-th coordinate is given by the easily obtainable expression

$$\dot{x}_\kappa = \sum_{\lambda=0}^{m(n-1)} {}^{(\lambda)}b_\kappa \varphi^{(\lambda+1)}(t) + \sum_{\lambda=0}^{mn-1} {}^{(\lambda)}c_\kappa \varphi^{(\lambda+2)}(t) \tag{41a}$$

Summarizing, it may be repeated that the solution for constant disturbance terms may be represented in a clear form: the determinants $D(p)$ are to be determined according to (26) and $\Delta_\kappa(p)$ with the coefficients ${}^{(\lambda)}b_\kappa$ according to the rule (39) whereas the coefficients ${}^{(\lambda)}c_\kappa$ and the function of time $\varphi(t)$ are given by (30) and (40), respectively.

3. Solution for Harmonic Disturbance Functions and Arbitrary Starting Conditions

If, aside from the starting phenomena, periodic permanent conditions also are of interest, it will be useful to write the disturbance functions in the form

$$F_\upsilon(t) = C_\upsilon e^{i\Omega_\upsilon t}$$

Ω_υ may be a real as well as a complex quantity. (36) yields as solution the relation

$$X_\kappa = \sum_{\upsilon=1}^{n} \sum_{\mu=0}^{mn-1} C_\upsilon \frac{A_{\upsilon\kappa}(p_\mu)}{D'(p_\mu)} \frac{e^{i\Omega_\upsilon t} - e^{p_\mu t}}{i\Omega_\upsilon - p_\mu} + \sum_{\mu=0}^{mn-1} \frac{D_{\kappa hom}(p_\mu)}{D'(p_\mu)} e^{p_\mu t} \tag{42}$$

and one recognizes that particular attention must be paid to cases where

$$i\Omega_\upsilon = p_\mu \tag{43}$$

Since all roots p_μ were assumed different from each other, this case will occur precisely for

$$i\Omega_\upsilon = p_o$$

If one writes (42) in the form

$$X_\kappa = \sum_{\upsilon=1}^{n} C_\upsilon \left\{ \frac{A_{\upsilon\kappa}(p_o)}{D'(p_o)} \frac{e^{i\Omega_\upsilon t} - e^{p_o t}}{i\Omega_\upsilon - p_o} + \sum_{\mu=1}^{mn-1} \frac{A_{\upsilon\kappa}(p_\mu)}{D'(p_\mu)} \frac{e^{i\Omega_\upsilon t} - e^{p_\mu t}}{i\Omega_\upsilon - p_\mu} \right\}$$

$$+ \sum_{\mu=0}^{mn-1} \frac{D_{\kappa hom}(p_\mu)}{D'(p_\mu)} e^{p_\mu t} \tag{42a}$$

and if now $i\Omega_\upsilon \longrightarrow p_o$, the first of the sum of (42a) assumes the indefinite form 0/0, whereas the rest remains certainly finite when the real parts $R(p_\mu)$ of the roots are less than zero.

Thus the further deliberations concern only the first expression for which one has to perform the limiting process

$$\lim_{i\Omega_\upsilon \longrightarrow p_o} \frac{e^{i\Omega_\upsilon t} - e^{p_o t}}{i\Omega_\upsilon - p_o} = te^{p_o t}$$

if one performs once more a limiting process with $t \longrightarrow \infty$

$$\lim_{t \to \infty} te^{p_o t} = \begin{cases} 0 & \text{for} \quad R(p_o)<0 \\ \infty & \text{for} \quad R(p_o) \geqq 0 \end{cases}$$

one recognizes that the solutions remain finite only if all real parts are $R(p_\mu)<0$.

Only under this assumption it seems logical to speak, in connection with (42), of a permanent condition for which then the relation

$$X_{\kappa Dau}** = \sum_{\upsilon=1}^{n} \sum_{\mu=0}^{mn-1} C_\upsilon \frac{A_{\upsilon\kappa}(p_\mu)}{D'(p_\mu)} \frac{e^{i\Omega_\upsilon t}}{i\Omega_\upsilon - p_\mu} \tag{46}$$

is valid.

From (31) follows for $p = i\Omega_\upsilon$ the relation

$$\frac{A_{\upsilon\kappa}(i\Omega_\upsilon)}{D(i\Omega_\upsilon)} = \sum_{\mu=0}^{mn-1} \frac{A_{\upsilon\kappa}(p_\mu)}{D'(p_\mu)} \frac{1}{i\Omega_\upsilon - p_\mu}$$

and one obtains

$$X_{\kappa Dau} = \sum_{\upsilon=1}^{n} e^{i\Omega_\upsilon t} C_\upsilon \frac{A_{\upsilon\kappa}(i\Omega_\upsilon)}{D(i\Omega_\upsilon)} \tag{47}$$

With the designations

$$C_\upsilon A_{\upsilon\kappa}(i\Omega_\upsilon) = \gamma_{\upsilon\kappa}(\Omega_\upsilon) + i\delta_{\upsilon\kappa}(\Omega_\upsilon) \tag{47a}$$

and

$$D(i\Omega_\upsilon) = \alpha_\upsilon(\Omega_\upsilon) + i\beta_\upsilon(\Omega_\upsilon) \tag{47b}$$

**Translator's Note: The subscript "Dau" is an abbreviation of the German word "Dauerzustand" signifying "permanent."

respectively, and the fact that real as well as imaginary part of (47) represent solutions, one obtains by transformation

$$X_{\kappa Dau} = \sum_{\upsilon=1}^{n} \left\{ \frac{\alpha_\upsilon \gamma_{\upsilon\kappa} + \beta_\upsilon \delta_{\upsilon\kappa}}{\alpha_\upsilon^2 + \beta_\upsilon^2} \cos \Omega_\upsilon t - \frac{\alpha_\upsilon \delta_{\upsilon\kappa} - \beta_\upsilon \gamma_{\upsilon\kappa}}{\alpha_\upsilon^2 + \beta_\upsilon^2} \sin \Omega_\upsilon t \right\} \tag{48a}$$

and

$$X_{\kappa Dau} = \sum_{\upsilon=1}^{n} \left\{ \frac{\alpha_\upsilon \gamma_{\upsilon\kappa} + \beta_\upsilon \delta_{\upsilon\kappa}}{\alpha_\upsilon^2 + \beta_\upsilon^2} \sin \Omega_\upsilon t + \frac{\alpha_\upsilon \delta_{\upsilon\kappa} - \beta_\upsilon \gamma_{\upsilon\kappa}}{\alpha_\upsilon^2 + \beta_\upsilon^2} \cos \Omega_\upsilon t \right\} \tag{48b}$$

respectively; however, for instance for (48a) one may write abbreviatedly

$$X_{\kappa Dau} = \sum_{\upsilon=1}^{n} V_{\upsilon\kappa} \cos(\Omega_\upsilon t + \epsilon_{\upsilon\kappa}) \tag{49}$$

with the amplitude function

$$V_{\upsilon\kappa} = \sqrt{\frac{\gamma_{\upsilon\kappa}^2 + \delta_{\upsilon\kappa}^2}{\alpha_v^2 + \beta_v^2}} \tag{50}$$

and the phase displacement

$$\epsilon_{\upsilon\kappa} = \arctan\left(\frac{\alpha_\upsilon \delta_{\upsilon\kappa} - \beta_\upsilon \gamma_{\upsilon\kappa}}{\alpha_\upsilon \gamma_{\upsilon\kappa} + \beta_\upsilon \delta_{\upsilon\kappa}}\right) \tag{51}$$

One has to consider that the expression takes into account the excitation of each individual coordinate with a different frequency. One then has to stress not only that it is possible to eliminate the roots of the characteristic equation but also that one may successfully include with the functions defined in (50) and (51) the influence of the υ-th excitation on the κ-th coordinate, and that one may obtain the total solution of the permanent equation by superposition of the partial solutions.

With these statements we shall regard the most general case of linear DES with constant coefficients and n degrees of freedom as closed.

VI. SOLUTION OF THE EQUATIONS OF MOTION DERIVED IN (IV)

The application of this method to the treatment of the prescribed differential equation system (17a), with the coefficients (17b) for the disturbance functions

$$\Delta S(\tau) = \text{Const} = \Delta S_{max}$$

and

$$\Delta S(\tau) = \Delta S_{max} e^{i\Omega\tau}$$

proceeds particularly simply.

Although the disturbance function ΔS_{max} = const. is not exactly realizable in practice, it is well to note its solution. It represents the upper limit for the controllability of an airplane by tabs and may serve as a comparative measure for control phenomena depending on time.

The explicit representation for periodic disturbance functions may be particularly valuable when information on resonance positions, amplitude functions, and phase displacements is required.

If one wants, on the other hand, to estimate the possible error which may appear with the problem discussed in Section V,2 – to replace disturbance functions dependent on time by suitable selection of constant disturbance functions C_υ – it is suitable to use disturbance functions linearily dependent on time.

1. Solution for $\Delta S(\tau)$ = Const. = ΔS_{max}

If one takes into consideration that by the selection of the steady initial state all starting conditions are $\Delta\alpha = \Delta\eta = \dot{\alpha} = \dot{\eta} = 0$ and divides the equations of (17a) by ΔS_{max}, using further on the designations $\frac{\Delta\alpha}{\Delta S_{max}} = X_2$ and $\frac{\Delta\eta}{\Delta S_{max}} = X_1$, one obtains according to (41) with $n = 2$ and $m = 2$ the relations

$$X_\kappa = \frac{\Delta_\kappa(0)}{D(0)} + \sum_{\lambda=0}^{2} {}^{(\lambda)}b_\kappa \varphi^{(\lambda)}(t) \tag{52}$$

and

$$\dot{X}_\kappa = \sum_{\lambda=0}^{2} {}^{(\lambda)}b_\kappa \varphi^{(\lambda+1)}(t) \tag{52a}$$

respectively.

For the determinant of the characteristic equation (26) one has to put with (23)

$$D(p) = \begin{vmatrix} \left({}^{(0)}a_{1\eta} + {}^{(1)}a_{1\eta}p + p^2\right) & \left({}^{(0)}a_{1\alpha} + {}^{(1)}a_{1\alpha}p + {}^{(2)}a_{1\alpha}p^2\right) \\ \left({}^{(0)}a_{2\eta} + 0 + 0\right) & \left({}^{(0)}a_{2\alpha} + {}^{(1)}a_{2\alpha}p + p^2\right) \end{vmatrix} = \sum_{\mu=0}^{4} a_\mu p^\mu \tag{53}$$

arranged according to powers of p, the relations

$$\left.\begin{aligned}
a_0 &= \begin{vmatrix} {}^{(0)}a_{1\eta} & {}^{(0)}a_{1\alpha} \\ {}^{(0)}a_{2\eta} & {}^{(0)}a_{2\alpha} \end{vmatrix} \qquad
a_1 = \begin{vmatrix} {}^{(0)}a_{1\eta} & {}^{(1)}a_{1\alpha} \\ {}^{(0)}a_{2\eta} & {}^{(1)}a_{2\alpha} \end{vmatrix} + \begin{vmatrix} {}^{(1)}a_{1\eta} & {}^{(0)}a_{1\alpha} \\ 0 & {}^{(0)}a_{2\alpha} \end{vmatrix} \\
a_2 &= \begin{vmatrix} {}^{(0)}a_{1\eta} & {}^{(2)}a_{1\alpha} \\ {}^{(0)}a_{2\eta} & 1 \end{vmatrix} + \begin{vmatrix} {}^{(1)}a_{1\eta} & {}^{(1)}a_{1\alpha} \\ 0 & {}^{(1)}a_{2\alpha} \end{vmatrix} + \begin{vmatrix} 1 & {}^{(0)}a_{1\alpha} \\ 0 & {}^{(0)}a_{2\alpha} \end{vmatrix} \\
a_3 &= \begin{vmatrix} {}^{(1)}a_{1\eta} & {}^{(2)}a_{1\alpha} \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & {}^{(1)}a_{1\alpha} \\ 0 & {}^{(1)}a_{2\alpha} \end{vmatrix} \qquad
a_4 = \begin{vmatrix} 1 & {}^{(2)}a_{1\alpha} \\ 0 & 1 \end{vmatrix} = 1
\end{aligned}\right\} \tag{53a}$$

are valid for the coefficients of the determinant (53).

The determinant $\Delta_\kappa(p)$ which one obtains according to the rule (39) by replacing the κ-th column of (53) by the disturbance terms C_υ, also arranging them according to powers of p, is

$$\Delta_\kappa(p) = \sum_{\upsilon=1}^{2} C_\upsilon A_{\upsilon\kappa}(p) = \sum_{\lambda=0}^{2} {}^{(\lambda)}b_\kappa p^\lambda \tag{54}$$

with the coefficients

$$\left.\begin{aligned}
{}^{(0)}b_1 &= \begin{vmatrix} C_1 & {}^{(0)}a_{1\alpha} \\ C_2 & {}^{(0)}a_{2\alpha} \end{vmatrix} = \Delta_1(0) \qquad & {}^{(0)}b_2 &= \begin{vmatrix} {}^{(0)}a_{1\eta} & C_1 \\ {}^{(0)}a_{2\eta} & C_2 \end{vmatrix} = \Delta_2(0) \\
{}^{(1)}b_1 &= \begin{vmatrix} C_1 & {}^{(1)}a_{1\alpha} \\ C_2 & {}^{(1)}a_{2\alpha} \end{vmatrix} & {}^{(1)}b_2 &= \begin{vmatrix} {}^{(1)}a_{1\eta} & C_1 \\ 0 & C_2 \end{vmatrix} \\
{}^{(2)}b_1 &= \begin{vmatrix} C_1 & {}^{(2)}a_{1\alpha} \\ C_2 & 1 \end{vmatrix} & {}^{(2)}b_2 &= \begin{vmatrix} 1 & C_1 \\ 0 & C_2 \end{vmatrix}
\end{aligned}\right\} \tag{54a}$$

With (40) the function $\varphi(t)$ is represented by the relation

$$\varphi(\tau) = \sum_{\mu=0}^{3} \frac{e^{p_\mu \tau}}{p_\mu D'(p_\mu)} \tag{55}$$

wherein p_μ are the simple roots of the characteristic equation (53) and with (33)

$$D'(p_\mu) = \prod_{\upsilon=0}^{3}{}^{(\mu)} (p_\mu - p_\upsilon) \tag{55a}$$

For the numerical utilization of (55) one has to distinguish three cases.

a) All roots real

b) Two real and one pair conjugate-complex roots

c) Two pairs conjugate-complex roots

a) All roots of the characteristic equation real

If all roots p_μ are real, the λ-th derivatives of (55) appearing in (52) and (52a) are

$$\varphi^{(\lambda)}(\tau) = \sum_{\mu=0}^{3} \frac{p_\mu^{\lambda-1} e^{p_\mu \tau}}{D'(p_\mu)} \tag{56}$$

and the solution (52) obtains immediately the form

$$x_\kappa = \frac{\Delta_\kappa(0)}{D(0)} + \sum_{\mu=0}^{3} d_\mu e^{p_\mu \tau} \tag{57}$$

with

$$d_\mu = \sum_{\lambda=0}^{2} {}^{(\lambda)}b_\kappa \frac{p_\mu^{\lambda-1}}{D'(p_\mu)} \tag{57a}$$

and

$$\dot{x}_\kappa = \sum_{\mu=0}^{3} d_\mu p_\mu e^{p_\mu \tau} \tag{57b}$$

respectively.

b) Two real and one pair conjugate-complex roots

Since here and also in the case c) in the expression for $\varphi^{(\lambda)}(\tau)$ complex quantities p_μ appear, the representation of the solutions (52) in terms of real quantities proceeds somewhat less simply. However, the presence of conjugate-complex roots signifies only that the solution contains periodic functions.

If one, therefore, splits off the two real roots p_0, p_1 and designates the pair of conjugate-complex roots by $p_{2,3} = \kappa_1 \pm i w_1$, one may write for (56) the expression

$$\varphi^{(\lambda)}(\tau) = {}^{(\lambda)}B_0 e^{p_0 \tau} + {}^{(\lambda)}C_0 e^{p_1 \tau} + \left\{ {}^{(\lambda)}B_1 \sin w_1 \tau + {}^{(\lambda)}C_1 \cos w_1 \tau \right\} e^{\kappa_1 \tau} \tag{58}$$

with

$$\varphi^{(0)}(\tau) = \varphi(\tau) \quad \text{and} \quad {}^{(0)}B_0 = B_0 \quad \text{etc.}$$

and

$$B_o = \frac{1}{p_o\left(p_o - p_1\right)\left[\left(p_o - \kappa_1\right)^2 + w_1^2\right]}$$

$$C_o = \frac{1}{p_1\left(p_1 - p_o\right)\left[\left(p_1 - \kappa_1\right)^2 + w_1^2\right]}$$

$$B_1 = \frac{1}{N}\left\{\kappa_1\left[\left(p_o - \kappa_1\right)\left(p_1 - \kappa_1\right) - w_1^2\right] + w_1^2\left(p_o + p_1 - 2\kappa_1\right)\right\}$$

$$C_1 = -\frac{1}{N}\left\{w_1\left[\left(p_o - \kappa_1\right)\left(p_1 - \kappa_1\right) - w_1^2\right] - \kappa_1 w_1\left(p_o + p_1 - 2\kappa_1\right)\right\}$$

$$N = w_1\left(\kappa_1^2 + w_1^2\right)\left[\left(p_o - \kappa_1\right)^2 + w_1^2\right]\left[\left(p_1 - \kappa_1\right)^2 + w_1^2\right]$$

$$^{(\lambda)}B_o = p_o^{\lambda}B_o \qquad ^{(\lambda)}B_1 = \kappa_1\,^{(\lambda-1)}B_1 - w_1\,^{(\lambda-1)}C_1$$

$$^{(\lambda)}C_o = p_1^{\lambda}C_o \qquad ^{(\lambda)}C_1 = w_1\,^{(\lambda-1)}B_1 + \kappa_1\,^{(\lambda-1)}C_1$$

If one now substitutes (58) into (52) and adds similar components of the solution, one obtains (58) in terms of real quantities

$$X_\kappa = \frac{\Delta_\kappa(0)}{D(0)} + \alpha_{o\kappa}e^{p_o\tau} + \beta_{o\kappa}e^{p_1\tau} + \left\{\alpha_{1\kappa}\sin w_1\tau + \beta_{1\kappa}\cos w_1\tau\right\}e^{\kappa_1\tau} \qquad (59)$$

with

$$\left.\begin{aligned} \alpha_{o\kappa} &= \sum_{\lambda=0}^{2} {}^{(\lambda)}b_{\kappa}{}^{(\lambda)}B_o & \alpha_{1\kappa} &= \sum_{\lambda=0}^{2} {}^{(\lambda)}b_{\kappa}{}^{(\lambda)}B_1 \\ \beta_{o\kappa} &= \sum_{\lambda=0}^{2} {}^{(\lambda)}b_{\kappa}{}^{(\lambda)}C_o & \beta_{1\kappa} &= \sum_{\lambda=0}^{2} {}^{(\lambda)}b_{\kappa}{}^{(\lambda)}C_1 \end{aligned}\right\} \tag{59a}$$

and by differentiation with respect to time the velocity

$$\dot{X}_{\kappa} = \bar{\alpha}_{o\kappa} e^{p_o\tau} + \bar{\beta}_{o\kappa} e^{p_1\tau} + \left\{\bar{\alpha}_{1\kappa}\sin w_1\tau + \bar{\beta}_{1\kappa}\cos w_1\tau\right\} e^{\kappa_1\tau} \tag{59b}$$

with

$$\left.\begin{aligned} \bar{\alpha}_{o\kappa} &= p_o\alpha_{o\kappa} & \bar{\alpha}_{1\kappa} &= \kappa_1\alpha_{1\kappa} - w_1\beta_{1\kappa} \\ \bar{\beta}_{o\kappa} &= p_1\beta_{o\kappa} & \bar{\beta}_{1\kappa} &= w_1\alpha_{1\kappa} + \kappa_1\beta_{1\kappa} \end{aligned}\right\} \tag{59c}$$

c) Two pairs of conjugate-complex roots

This case is solved, if one puts $p_{o,1} = \kappa_o \pm iw_o$ and $p_{2,3} = \kappa_1 \pm iw_1$, with the relation

$$\varphi^{(\lambda)}(\tau) = \sum_{\upsilon=0}^{1} \left\{{}^{(\lambda)}B_{\upsilon}\sin w_{\upsilon}\tau + {}^{(\lambda)}C_{\upsilon}\cos w_{\upsilon}\tau\right\} e^{\kappa_{\upsilon}\tau} \tag{60}$$

with

$$
\left.
\begin{aligned}
B_0 &= \frac{w_1\left(\kappa_1^2 + w_1^2\right)}{N}\left\{\kappa_0\left[\left(\kappa_0 - \kappa_1\right)^2 - \left(w_0^2 - w_1^2\right)\right] - 2w_0^2\left(\kappa_0 - \kappa_1\right)\right\} \\
C_0 &= -\frac{w_1\left(\kappa_1^2 + w_1^2\right)}{N}\left\{w_0\left[\left(\kappa_0 - \kappa_1\right)^2 - \left(w_0^2 - w_1^2\right)\right] + 2\kappa_0 w_0\left(\kappa_0 - \kappa_1\right)\right\} \\
B_1 &= \frac{w_0\left(\kappa_0^2 + w_0^2\right)}{N}\left\{\kappa_1\left[\left(\kappa_0 - \kappa_1\right)^2 + \left(w_0^2 - w_1^2\right)\right] + 2w_1^2\left(\kappa_0 - \kappa_1\right)\right\} \\
C_1 &= -\frac{w_0\left(\kappa_0^2 - w_0^2\right)}{N}\left\{w_1\left[\left(\kappa_0 - \kappa_1\right)^2 + \left(w_0^2 - w_1^2\right)\right] - 2\kappa_1 w_1\left(\kappa_0 - \kappa_1\right)\right\} \\
N &= D(0)w_0 w_1\left\{\left(\kappa_0 - \kappa_1\right)^2 + \left(w_0 - w_1\right)^2\right\}\left\{\left(\kappa_0 - \kappa_1\right)^2 + \left(w_0 + w_1\right)^2\right\} \\
{}^{(\lambda)}B_\upsilon &= \kappa_\upsilon {}^{(\lambda-1)}B_\upsilon - w_\upsilon {}^{(\lambda-1)}C_\upsilon \\
{}^{(\lambda)}C_\upsilon &= w_\upsilon {}^{(\lambda-1)}B_\upsilon + \kappa_\upsilon {}^{(\lambda-1)}C_\upsilon
\end{aligned}
\right\}
\tag{60a}
$$

If one introduces here also (60) into (52) and forms the sum of similar terms, there appears here as the type of solution the form

$$
X_\kappa = \frac{\Delta_\kappa(0)}{D(0)} + \sum_{\upsilon=0}^{1}\left\{\alpha_{\upsilon\kappa}\sin w_\upsilon\tau + \beta_{\upsilon\kappa}\cos w_\upsilon\tau\right\}e^{\kappa_\upsilon\tau} \tag{61}
$$

with

$$
\alpha_{\upsilon\kappa} = \sum_{\lambda=0}^{2} {}^{(\lambda)}b_\kappa {}^{(\lambda)}B_\upsilon \qquad \beta_{\upsilon\kappa} = \sum_{\lambda=0}^{2} {}^{(\lambda)}b_\kappa {}^{(\lambda)}C_\upsilon \tag{61a}
$$

and for the derivative

$$\dot{X}_\kappa = \sum_{\upsilon=0}^{1} \left\{\bar{\alpha}_{\upsilon\kappa} \sin w_\upsilon \tau + \bar{\beta}_{\upsilon\kappa} \cos w_\upsilon \tau\right\} e^{\kappa_\upsilon \tau} \tag{61b}$$

with

$$\left.\begin{aligned} \bar{\alpha}_{\upsilon\kappa} &= \kappa_\upsilon \alpha_{\upsilon\kappa} - w_\upsilon \beta_{\upsilon\kappa} \\ \bar{\beta}_{\upsilon\kappa} &= w_\upsilon \alpha_{\upsilon\kappa} + \kappa_\upsilon \beta_{\upsilon\kappa} \end{aligned}\right\} \tag{61c}$$

2. Solution for Harmonic Disturbance Function $\Delta S(\tau) = \Delta S_{max} e^{i\Omega\tau}$

The solution of this special case is suitable for demonstrating the behavior of airplane controls by control surfaces and tabs for periodic excitations as they may be caused, for instance, by engine vibrations, fuselage-bending oscillations, or torsional oscillations of the stabilizing surfaces. However, one will have to pay special attention to the permanent forced oscillating state in comparison with the starting conditions.

Noting the fact that the function $e^{i\Omega\tau}$ which is dependent on time is independent of any summation, one obtains with (47) and the relation (54) as solution the permanent equation

$$X_{\kappa Dau} = e^{i\Omega\tau} \frac{\Delta_\kappa(i\Omega)}{D(i\Omega)} \tag{62}$$

the values of the determinants $\Delta_\kappa(p)$ and $D(p)$ have to be determined according to (54) and (53) for purely imaginary arguments $p = i\Omega$.

Consequently it is useful to choose for these complex expressions the form

$$\left.\begin{aligned} \Delta_\kappa(i\Omega) &= \gamma_\kappa(\Omega) + i\delta_\kappa(\Omega) \\ D(i\Omega) &= \zeta(\Omega) + i\psi(\Omega) \end{aligned}\right\} \qquad (63)$$

with the real functions of Ω

$$\gamma_\kappa = {}^{(0)}b_\kappa - {}^{(2)}b_\kappa\Omega^2$$

$$\delta_\kappa = {}^{(1)}b_\kappa\Omega$$

$$\zeta = a_o - a_2\Omega^2 - \Omega^4$$

$$\psi = a_1\Omega - a_3\Omega^3$$

the coefficients of which are given with (53a) and (54a), respectively.

If the real part of (62) is to be regarded as solution, there results because of (38)

$$X_{\kappa Dau} = V_\kappa \cos(\Omega\tau + \epsilon_\kappa) \qquad (64)$$

with the amplitude function

$$V_\kappa = \sqrt{\frac{{\gamma_\kappa}^2 + {\delta_\kappa}^2}{\zeta^2 + \psi^2}} \qquad (64a)$$

and the phase displacement

$$\epsilon_\kappa = \text{arc tan}\left\{\frac{\zeta\delta_\kappa - \psi\gamma_\kappa}{\zeta\gamma_\kappa + \psi\delta_\kappa}\right\} \qquad (64b)$$

3. Solution for Linearily Dependent Disturbance Function $\Delta S(\tau) = A\tau$

As already mentioned at the outset, this solution may serve for comparison of instantaneous and time-dependent sudden deflection of the tabs, if one superposes the two phenomena

$$\left.\begin{array}{lll} \Delta S(\tau) = a\tau & \text{for} & 0 \leqq \tau < \infty \\ \text{and} & & \\ \Delta S(\tau) = \begin{cases} 0 & \text{for} \quad 0 \leqq \tau < \tau_1 \\ a(\tau - \tau_1) & \text{for} \quad \tau_1 \leqq \tau < \infty \end{cases} & & \end{array}\right\} \tag{65}$$

As resultant deflection phenomenon one obtains for instance the disturbance function represented in figure 6. Particular interest will be attached to that deflection phenomenon which characterizes the maximum velocity of deflection of the tabs possible in practice and is given by

$$a = \frac{\Delta S_{max}}{\tau_1} \tag{65a}$$

To attain the solution one will go back to the general form of equation (36).

With the fact already stressed in Section VI, 1 that all starting conditions are zero, there follows with

$$F_\upsilon(\tau) = C_\upsilon \frac{\Delta S_{max}}{\tau_1}\tau$$

after performance of the integration for the ratio $\frac{\Delta\alpha}{\Delta S_{max}} = X_1$

and $\frac{\Delta\eta}{\Delta S_{max}} = X_2$, respectively,

$$X_\kappa = \frac{1}{\tau_1}\sum_{\upsilon=1}^{2}\sum_{\mu=0}^{3}\frac{C_\upsilon A_{\upsilon\kappa}(p_\mu)}{{p_\mu}^2 D'(p_\mu)}\left\{e^{p_\mu\tau} - p_\mu\tau - 1\right\} \tag{66}$$

With the relation (64), however, one can write

$$X_\kappa = \frac{1}{\tau_1}\left\{\frac{\Delta_\kappa(0)}{D(0)}\tau + \sum_{\mu=0}^{3}\frac{\Delta_\kappa(p_\mu)}{p_\mu{}^2 D'(p_\mu)}e^{p_\mu \tau} - \sum_{\mu=0}^{3}\frac{\Delta_\kappa(p_\mu)}{p_\mu{}^2 D'(p_\mu)}\right\} \tag{66a}$$

or, after according rearrangement and introduction of the notation in powers for

$$\left.\begin{aligned} X_\kappa = \frac{1}{\tau_1}\Bigg\{&\frac{\Delta_\kappa(0)}{D(0)}\tau + \sum_{\mu=0}^{3}{}^{(0)}b_\kappa\left[\frac{e^{p_\mu \tau}}{p_\mu{}^2 D'(p_\mu)} - \frac{1}{p_\mu{}^2 D'(p_\mu)}\right] + \sum_{\mu=0}^{3}{}^{(1)}b_\kappa\frac{e^{p_\mu \tau}}{p_\mu D'(p_\mu)} \\ &+ \sum_{\mu=0}^{3}{}^{(2)}b_\kappa\frac{e^{p_\mu \tau}}{D'(p_\mu)} - \sum_{\mu=0}^{3}\sum_{\lambda=1}^{2}\frac{{}^{(\lambda)}b_\kappa}{p_\mu{}^2 D'(p_\mu)}\Bigg\}\end{aligned}\right\} \tag{66b}$$

If one forms from (55) the integral

$$\int_0^\tau \varphi(\xi)d\xi = \sum_{\mu=0}^{3}\left\{\frac{e^{p_\mu \tau}}{p_\mu{}^2 D'(p_\mu)} - \frac{1}{p_\mu{}^2 D'(p_\mu)}\right\} \tag{67}$$

and introduces the designation $\int_0^\tau \varphi(\xi)d\xi = \varphi^{(-1)}(\tau)$ one may write for (66), in analogy to (52), also

$$X_\kappa = \frac{1}{\tau_1}\left\{\frac{\Delta_\kappa(0)}{D(0)} + \sum_{\lambda=0}^{2}{}^{(\lambda)}b_\kappa \varphi^{(\lambda-1)}(\tau) - \sum_{\mu=0}^{3}\sum_{\lambda=1}^{2}\frac{{}^{(\lambda)}b_\kappa}{p_\mu{}^2 D'(p_\mu)}\right\} \tag{66c}$$

Corresponding to the three combinations of the roots of the characteristic equation (53) treated in Section VI, 1 these three possibilities could be investigated here also.

Let it suffice to point out the procedure for the case of two pairs of conjugate-complex roots. As analogon to (60) one may select the expression

$$\int_0^{\tau}\varphi(\xi)d\xi = \sum_{\upsilon=0}^{1}\left\{\left[{}^{(-1)}B_{\upsilon}\sin w_{\upsilon}\tau + {}^{(-1)}C_{\upsilon}\cos w_{\upsilon}\tau\right]e^{\kappa_{\upsilon}\tau} - {}^{(-1)}C_{\upsilon}\right\} \tag{67a}$$

with

$$^{(-1)}B_{\upsilon} = \frac{\kappa_{\upsilon}B_{\upsilon} + w_{\upsilon}C_{\upsilon}}{\kappa_{\upsilon}^2 + w_{\upsilon}^2}$$

$$^{(-1)}C_{\upsilon} = \frac{-w_{\upsilon}B_{\upsilon} + \kappa_{\upsilon}C_{\upsilon}}{\kappa_{\upsilon}^2 + w_{\upsilon}^2}$$

with the B_{υ} and C_{υ} to be calculated according to (60b). With (67a) and (60), (66c) is transformed into the form

$$\left.\begin{aligned} X_{\kappa} = \frac{1}{\tau_1}\Bigg\{&\frac{\Delta_{\kappa}(0)}{D(0)}\tau + \sum_{\upsilon=0}^{1}\left[\alpha_{\upsilon\kappa}\sin w_{\upsilon}\tau + \beta_{\upsilon\kappa}\cos w_{\upsilon}\tau\right]e^{\kappa_{\upsilon}\tau} \\ &-\left[{}^{(-1)}C_0 + {}^{(-1)}C_1 + \sum_{\mu=0}^{3}\sum_{\lambda=1}^{2}\frac{{}^{(\lambda)}b_{\kappa}}{p_{\mu}{}^{2}D'(p_{\mu})}\right]\Bigg\}\end{aligned}\right\} \tag{68}$$

with

$$\left.\begin{aligned}\alpha_{\upsilon\kappa} &= \sum_{\lambda=0}^{2}{}^{(\lambda)}b_{\kappa}{}^{(\lambda-1)}B_{\upsilon} \\ \beta_{\upsilon\kappa} &= \sum_{\lambda=0}^{2}{}^{(\lambda)}b_{\kappa}{}^{(\lambda-1)}C_{\upsilon}\end{aligned}\right\} \tag{68a}$$

If one introduces into (68) according to rule (65) the argument $(\tau - \tau_1)$ instead of the independent variable τ and superposes the solutions thus obtained, the solution for the deflection phenomenon sketched in figure 6 assumes the shape

$$\left[X_\kappa\right]_\tau - \left[X_\kappa\right]_{\tau-\tau_1} = \frac{\Delta_\kappa(0)}{D(0)} + \sum_{\upsilon=0}^{1}\left\{\bar{\alpha}_{\upsilon\kappa}\sin w_\upsilon\tau + \bar{\beta}_{\upsilon\kappa}\cos w_\upsilon\tau\right\} e^{\kappa_\upsilon\tau} \tag{69}$$

with

$$\left.\begin{aligned}
\bar{\alpha}_{\upsilon\kappa} &= \frac{1}{\tau_1}\left\{\alpha_{\upsilon\kappa}\left(1 - e^{-\kappa_\upsilon\tau_1}\cos w_\upsilon\tau_1\right) + \beta_{\upsilon\kappa}e^{-\kappa_\upsilon\tau_1}\sin w_\upsilon\tau_1\right\} \\
\bar{\beta}_{\upsilon\kappa} &= \frac{1}{\tau_1}\left\{\alpha_{\upsilon\kappa}e^{-\kappa_\upsilon\tau_1}\sin w_\upsilon\tau_1 + \beta_{\upsilon\kappa}\left(1 - e^{-\kappa_\upsilon\tau_1}\cos w_\upsilon\tau_1\right)\right\}
\end{aligned}\right\} \tag{69a}$$

whereas for the derivative the relation

$$\dot{X}_\kappa = \sum_{\upsilon=0}^{1}\left\{\bar{\bar{\alpha}}_{\upsilon\kappa}\sin w_\upsilon\tau + \bar{\bar{\beta}}_{\upsilon\kappa}\cos w_\upsilon\tau\right\} e^{\kappa_\upsilon\tau} \tag{69b}$$

with

$$\left.\begin{aligned}
\bar{\bar{\alpha}}_{\upsilon\kappa} &= \kappa_\upsilon\bar{\alpha}_{\upsilon\kappa} - w_\upsilon\bar{\beta}_{\upsilon\kappa} \\
\bar{\bar{\beta}}_{\upsilon\kappa} &= w_\upsilon\bar{\alpha}_{\upsilon\kappa} + \kappa_\upsilon\bar{\beta}_{\upsilon\kappa}
\end{aligned}\right\} \tag{69c}$$

exists and one must note that (69) is valid only in the interval $\tau_1 < \tau < \infty$.

The first partial report may be closed with these theoretical expositions. In the second partial report the results of the present report will be applied to large aircraft controlled by tabs.

VII. SUMMARY

The present report is intended as a contribution toward the clarification of problems arising for longitudinal motions of airplanes which are controlled not by direct control surface activation but indirectly by tabs.

After a general derivation of the equations of motion for the four degrees of freedom X_g, Z_g, ϑ, and η and taking into consideration that the control surfaces are not weight-compensated, permissible assumptions are introduced so that the system of four degrees of freedom is reduced to a system with the two degrees of freedom α and η. By the substitution $\tau = t\sqrt{q}$ a representation of the equations of motion which is independent of the dynamic pressure becomes possible inasmuch as one regards the dynamic pressure ratio q_H/q and the downwash factor $\partial\Delta/\partial\alpha$ as constant.

While stability and oscillation conditions may be discussed already with a knowledge of the solution of homogeneous differential equation systems, one must know also the solutions of inhomogeneous differential equation systems for the investigation of starting phenomena. Since the required mathematical expedients appear suitable not only for the solution of the prescribed problem but also for the analytical description of similar problems, they are treated in the most general formulation for a linear differential equation system of n degrees of freedom with constant coefficients, and arbitrary starting conditions and disturbance functions as far as the latter are integrable.

From the general solution special solutions for constant and periodic disturbance functions are developed.

With the definition of a function $\varphi(t)$ which depends only on the roots of the characteristic equation one may obtain for constant disturbance functions a solution of a very clearly arranged form; it consists of the three constituents: constant, inhomogeneous, and homogeneous part of the solution.

For periodic excitations one determines first the conditions which are required to make a permanent state possible. Then one can demonstrate that amplitude and phase-displacement functions exist which include the

influence of the υ-th excitation on the κ-th coordinate. The total solution of the permanent equations is then obtained by superposition of the partial solutions.

These results yield, applied to the prescribed problem, the desired solutions; with regard to numerical calculation real representation of the functions is particularly emphasized.

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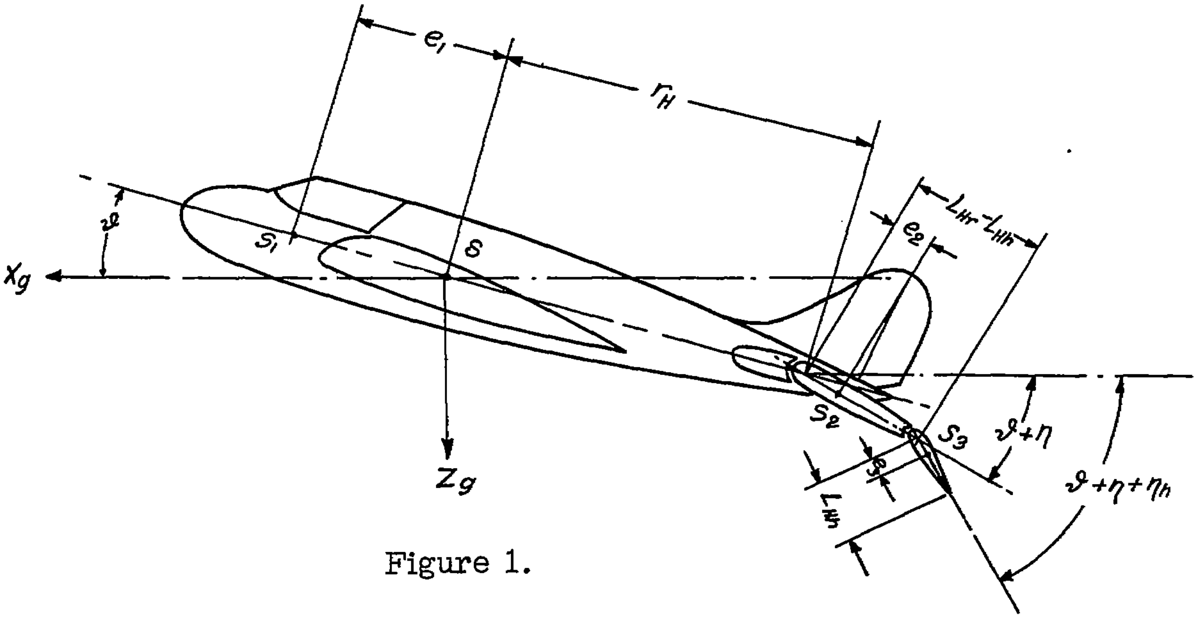

Figure 1.

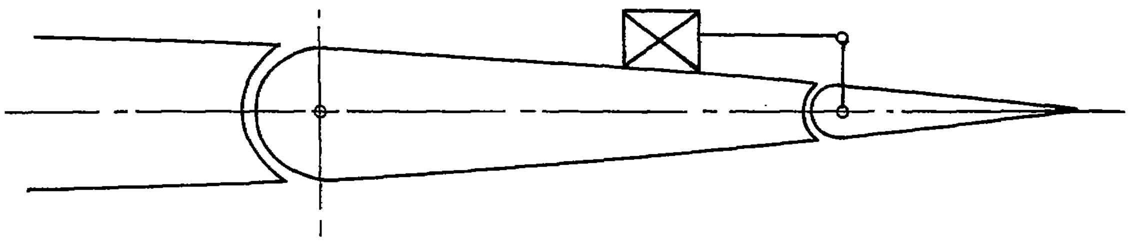

Figure 2.- Direct tab control.

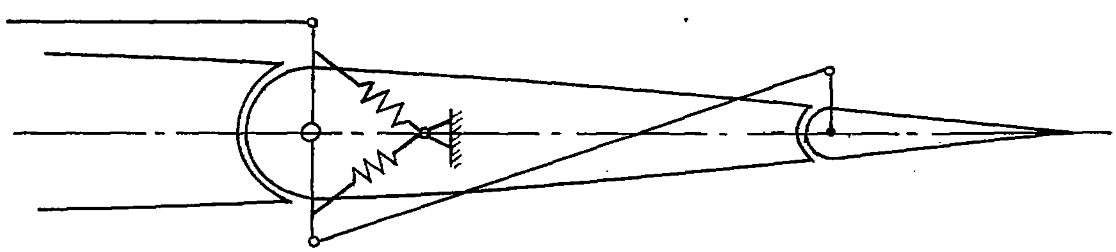

Figure 3.- Indirect tab control.

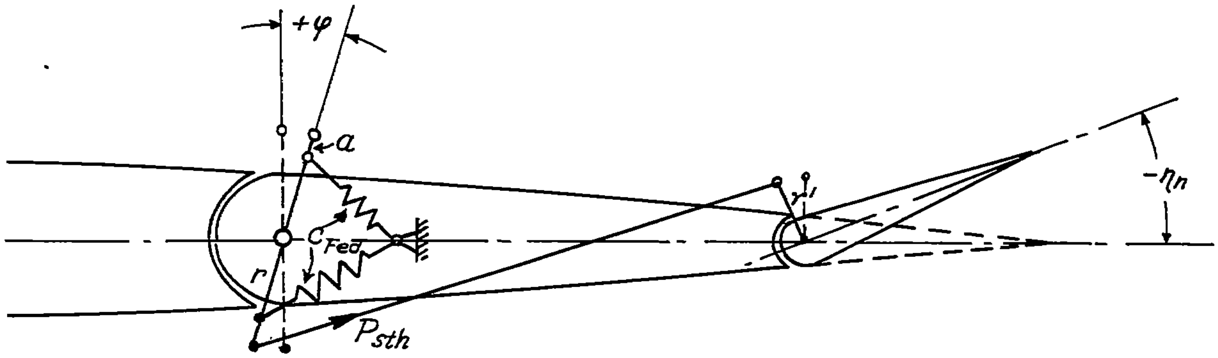

Figure 4.- Sketch of the system.

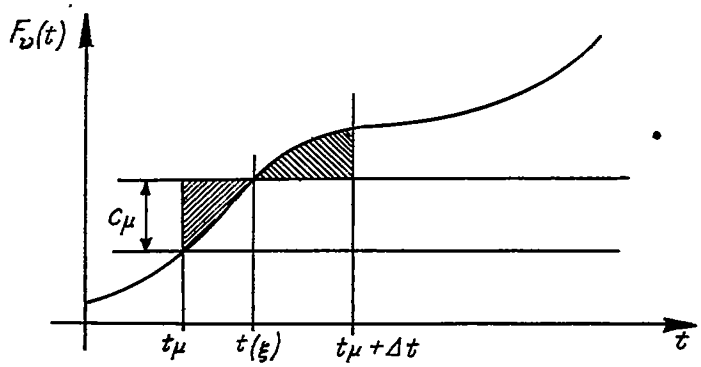

Figure 5.

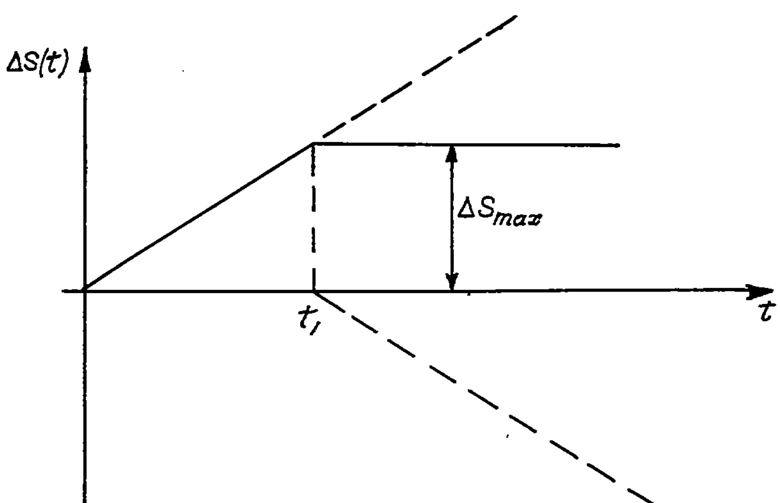

Figure 6.

NACA-Langley - 9-26-49 - 775